Yo-Yo Physics:

An Engineer's Notebook

Radius of Gyration

Monograph I
in a series

Penciled and
Published by:

Captain Yo

CONTENTS

INTRODUCTION

AN "ENGINEER'S NOTEBOOK"? WELL, NOT EXACTLY. EVERYTHING HERE IS REWORKED FROM ROUGH NOTES, ROUGHER SKETCHES, SCRATCH SHEETS, AND WHATNOT INTO THIS FORM.

A YO-YO DESIGNED, MANUFACTURED AND IN A PLAYER'S HAND "WINDS UP" WITH PHYSICAL CHARACTERISTICS WHICH, WITH THE PLAYER'S SKILL AND STRING EFFECTS, DETERMINE ITS PERFORMANCE. CERTAIN OF THE CHARACTERISTICS ARE REAL AND DIRECTLY MEASURED, OTHERS ARE CONCEPTUAL (IMAGINARY) BUT CAN BE RECKONED USING COMMON PHYSICS AND MATHEMATICS TECHNIQUES. THE MOMENT OF INERTIA AND THE RADIUS OF GYRATION ARE SPECIFIC TARGETS IN THIS MONOGRAPH.

I'M STRONGLY INDEBTED TO TOM KUHN, BRAD COUNTRYMAN, AND DALE OLIVER FOR PROFESSIONAL ASSIGNMENTS OVER THE PAST TEN YEARS WHERE THESE TECHNIQUES AND OTHERS HAVE BEEN OF SOME USE. PERMISSION TO USE NON-PROPRIETARY INFORMATION FROM CERTAIN OF THEIR PRODUCTS IS GRATEFULLY ACKNOWLEDGED - AS ARE THEIR MAJOR CONTRIBUTIONS TO THE YO-YO ART AND CULTURE OVER THE YEARS.

Don Watson
2/12/2000

RADIUS OF GYRATION

IN "AROUND THE WORLD" A YO-YO OF COMMON WEIGHT OR MASS (M, 50 grams) FLIES ABOUT THE PLAYER'S HAND IN A CIRCULAR PATH AT A RADIUS OF GYRATION k_o (1 meter, A TYPICAL STRING LENGTH). IN THIS SITUATION THE MOMENT OF INERTIA (I, kilogram-meters^2) FOR THE YO-YO IS:

$$I = Mk_o^2 = 50 \times 10^{-3} \text{kg} \cdot (1\,\text{m})^2$$

$$\underline{\underline{I = 50 \times 10^{-3}}} \text{kg-m}^2$$

M, k_o, AND I, ONCE DEFINED, DO NOT CHANGE; THE NUMBERS EXIST WHETHER THE YO-YO FLIES (NO MATTER HOW QUICKLY OR SLOWLY) OR SITS AT REST.

IN ANOTHER SITUATION WHERE THE YO-YO SPINS ABOUT ITS INTERNAL AXIAL CENTER THE MASS (M) REMAINS KNOWN, BUT THE RADIUS OF GYRATION (k_o) IS IMAGINARY. THE FOLLOWING TECHNIQUE ACCURATELY YIELDS k_o FOR A YO-YO IN HAND OR FOR A YO-YO OF PROPOSED DESIGN:

1. DETERMINE THE DIMENSIONS FOR THE YO-YO PROFILE AND RESOLVE THAT PROFILE INTO CONSTITUENT AREA ELEMENTS.

2. VISUALIZE THE SOLID OF REVOLUTION THAT CAN BE GENERATED FOR EACH ELEMENT;

FIND THE VOLUME OF EACH SOLID AND, FROM DENSITY OF THE MATERIAL, THE WEIGHT OF MASS (M) OF EACH.

3. USING THE DIMENSIONS OF EACH AREA ELEMENT AND THE MASS OF ITS SOLID OF REVOLUTION, CALCULATE THE MOMENT OF INERTIA (I) FOR EACH SOLID.

4. SUMMARIZE THE TOTAL MASS AND THE TOTAL MOMENT OF INERTIA FOR THE YO-YO FROM ITS CONSTITUENT VALUES.

5. FINALLY, CALCULATE k_o FOR THE YO-YO FROM ITS NOW KNOWN TOTAL M AND TOTAL I:

$$k_o = \left(\frac{I}{M}\right)^{\frac{1}{2}} m$$

A TYPICAL YO-YO MIGHT HAVE A MASS OF 50×10^{-3} kg AND A MOMENT OF INERTIA OF 20000×10^{-9} kg-m^2 TO YIELD:

$$k_o = \left(\frac{20000 \times 10^{-9}}{50 \times 10^{-3}}\right)^{\frac{1}{2}} = (400 \times 10^{-6})^{\frac{1}{2}}$$

$$= 20.00 \times 10^{-3} m / 25.4 \times 10^{-3} m/in^{*}$$

$$\underline{\underline{k_o = 0.787 \, in}}$$

* IN THE STUDIES FOLLOWING, THE ENGLISH SYSTEM (inches, pounds) IS USED WITH THE INTERNATIONAL SYSTEM (meters, grams); THE LATTER COMMONLY STANDARD IN THE SCIENTIFIC COMMUNITY AND ITS REFERENCES, THE FORMER STILL COMMON IN U.S. ENGINEERING.

LEVERAGE: STRING TO YO-YO

IN ITS PATH ABOUT THE PLAYER'S HAND IN "LOOP THE LOOP" FLIGHT, THE STRING HAS BEEN ALMOST FULLY REWOUND, THE YO-YO EXECUTES A LATERAL HALF-FLIP, AND THE SPIN ABOUT ITS OWN AXIS CEASES THEN RESTARTS AS IT MOVES UNDER THE HAND INTO THE NEXT LOOP (SEE AT RIGHT). THE FORCE OR TENSION (T) HERE IS EXERTED BY THE HAND TO ACCELERATE THE SPIN VELOCITY (ω) AGAINST THE ROTATIONAL INERTIA OF THE YO-YO.

THE TENSION (T) ACTS AT THE WOUND STRING RADIUS (r_s, 0.50 in) AGAINST THE MOMENT OF INERTIA (I, 20000×10^{-9} kg-m^2) OF THE MASS (M, 50×10^{-3} kg) ACTING AT THE (LARGER) RADIUS OF GYRATION (k_o, 0.80 in).

THE TENSION (T) IS NOT FULLY EFFECTIVE IN ACCELERATING THE SPIN VELOCITY (ω); IT IS REDUCED BY THE LEVERAGE (L%), STRING TO YO-YO:

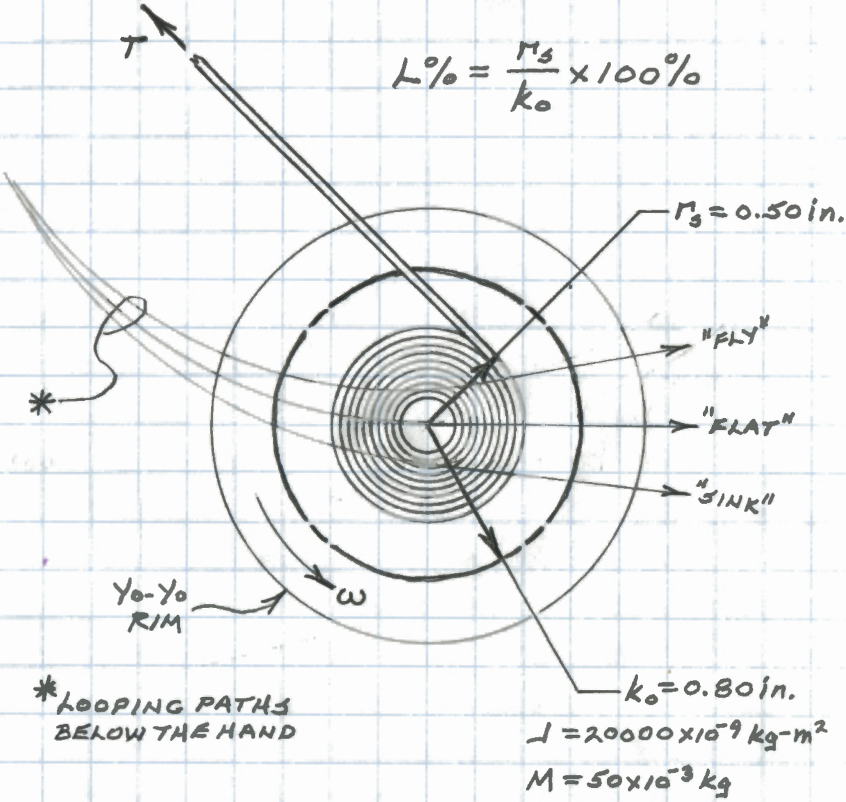
$$L\% = \frac{r_s}{k_o} \times 100\% = \frac{0.50}{0.80} \times 100\%$$

$$\underline{\underline{L\% = 63\%}}$$

ASSUME THE SPIN ACCELERATION AT THIS LEVERAGE IS JUST ENOUGH TO ALLOW THE YO-YO TO LOOP OUT "FLAT" OR NEARLY SO. IT

LEVERAGE:

$$L\% = \frac{r_s}{k_o} \times 100\%$$



*LOOPING PATHS BELOW THE HAND

LOOPING DYNAMICS

NOTE: IN THIS ACTION PICTURE - WHILE ALL THAT "FLYIN'", "FLATTIN'", OR "SINKIN'" IS GOING ON, THE MASS (M), MOMENT OF INERTIA (J), AND RADIUS OF GYRATION (k_o) ARE BUILT-IN PHYSICAL CONSTANTS. THEY "SET THE STAGE" FOR THE YO-YO PERFORMANCE, WHILE THE PHYSICAL CHARACTERISTICS OF THE STRING AND GAP IN WHICH IT WINDS PROVIDE (WITH THE PLAYER) THE FINAL SHOW.

FOLLOWS THAT REDUCING THE WOUND STRING RADIUS (r_s) WITH A WIDER STRING GAP OR BY USING SHORTER (OR THINNER) STRING RESULTS IN LOWER LEVERAGE AND LESS ACCELERATION. THUS, AT THIS CRITICAL TIME, THE STRING UNWINDS AT A LESSER RATE AND THE LOOPING PATH DOES NOT FLATTEN OUT; THE YO-YO WILL "FLY". USING A NARROWER STRING GAP, LONGER OR FATTER STRING, HAS THE REVERSE EFFECT, ALLOWING THE STRING TO UNWIND MORE RAPIDLY; THE YO-YO THEN WILL "SINK".

IN ANY YO-YO DESIGN, EXISTING OR PROJECTED, THE LIKELY LEVERAGE (L%) MAY BE USED TO GOOD ADVANTAGE AS A "FIGURE OF MERIT" TO EXPLAIN OR PREDICT THE QUALITY OF LOOPING ACTION. THE PHYSICAL CHARACTERISTICS OF THE YO-YO ARE NEEDED HERE, ESPECIALLY THE RADIUS OF GYRATION (k_o).

PLAYERS LOOK FOR MORE IN A "GOOD LOOPER" THAN JUST THE YO-YO'S TENDENCY TO "FLY" OR "SINK". IF EITHER OF THOSE CHARACTERISTICS ARE STRONGLY EVIDENT, THE YO-YO – IN THE ABSENCE OF SOME OTHER SAVING GRACE – MAY BE PUT ASIDE. A YO-YO NEED NOT BE A GOOD "LOOPER" TO BE AN OTHERWISE GREAT PERFORMER.

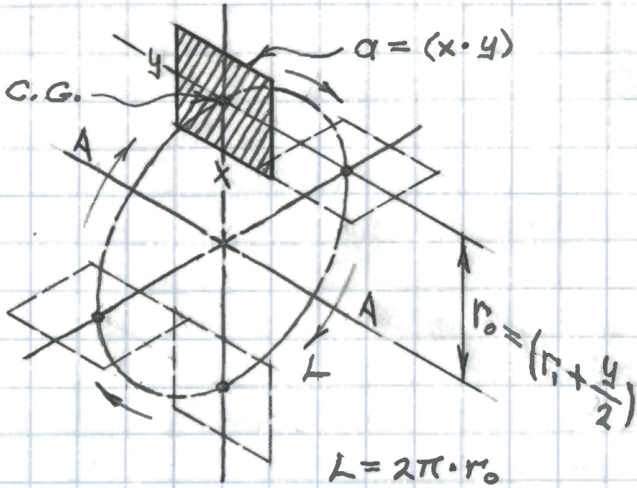
SOLID OF REVOLUTION VOLUME

THE CENTER OF GRAVITY (C.G.) OF A DEFINED SURFACE IS THAT POINT IN THE SURFACE WHERE SUSPENSION OF THE SURFACE FINDS IT PERFECTLY BALANCED IN ALL POSITIONS.

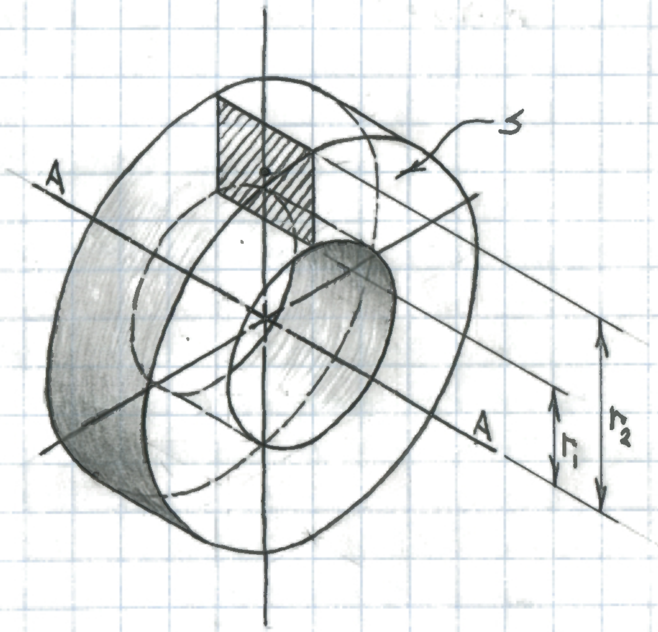
ON THE FOLLOWING PAGE, THE VOLUME (V_s) OF THE SOLID BODY (S) FORMED BY REVOLVING THE SURFACE (a) ABOUT THE AXIS (A-A) EQUALS THE AREA ($x \cdot y$) MULTIPLIED BY THE LENGTH (L) OF THE PATH OF ITS CENTER OF GRAVITY. THE DEFINED SURFACE MAY BE OF ANY (FLAT) SHAPE, BUT MUST LIE WHOLLY ON ONE SIDE OF AND IN A PLANE WITH THE AXIS OF REVOLUTION.

THE PROFILE OF A YO-YO CAN BE SEPARATED INTO AREA ELEMENTS. THE SOLID OF REVOLUTION VOLUME AND WEIGHT FOR EACH ARE THEN CALCULATED; THEN THE MOMENT OF INERTIA FOR EACH IS DETERMINED. FINALLY, THE TOTAL WEIGHT AND THE COMBINED MOMENT OF INERTIA OF THE YO-YO ARE USED TO FIND THE RADIUS OF GYRATION.

THE YO-YO IS ITSELF AN ULTIMATE "SOLID OF REVOLUTION" IN TWO USUALLY IDENTICAL HALVES ON A COMMON AXLE.

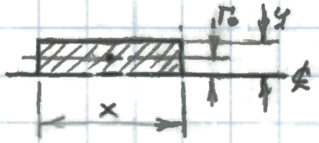
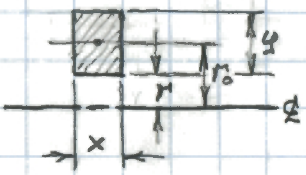
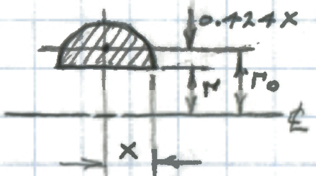
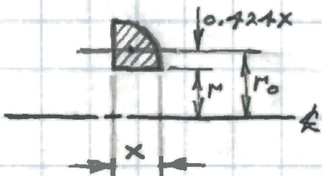
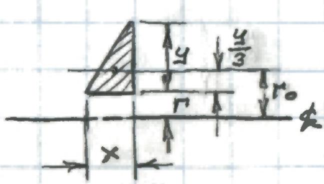


$$L = 2\pi \cdot r_0$$



$$V_S = (x \cdot y) \cdot 2\pi \cdot \left(r_1 + \frac{y}{2}\right)$$

SOLID OF REVOLUTION
VOLUME

AREA ELEMENT		SOLID OF REVOLUTION VOLUME
I.	 $a = x \cdot y$	$r_o = \frac{y}{2}$ $V = (x \cdot y) \cdot 2\pi \cdot \frac{y}{2}$
II.	 $a = x \cdot y$	$r_o = (r + \frac{y}{2})$ $V = (x \cdot y) \cdot 2\pi \cdot (r + \frac{y}{2})$
III.	 $a = \frac{\pi x^2}{2}$	$r_o = (r + 0.424x)$ $V = (\frac{\pi x^2}{2}) \cdot 2\pi \cdot (r + 0.424x)$
IV.	 $a = \frac{\pi x^2}{4}$	$r_o = (r + 0.424x)$ $V = (\frac{\pi x^2}{4}) \cdot 2\pi \cdot (r + 0.424x)$
V.	 $a = \frac{x \cdot y}{2}$	$r_o = (r + \frac{y}{3})$ $V = (\frac{x \cdot y}{2}) \cdot 2\pi \cdot (r + \frac{y}{3})$

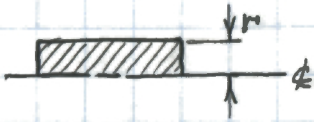
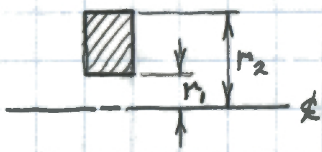
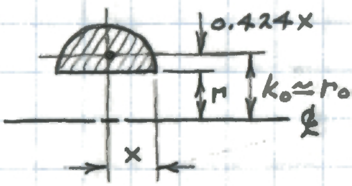
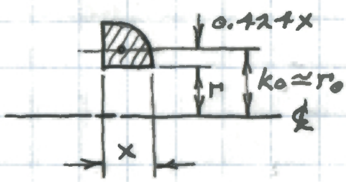
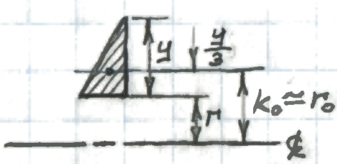
SOLID OF REVOLUTION
MOMENT OF INERTIA

THE MASS (M) OF A ROTATING BODY CAN BE CONSIDERED CONCENTRATED AT THE RADIUS OF GYRATION (k_o); AT THAT RADIUS THE MOMENT OF INERTIA (I) IS DETERMINED ACCURATELY TO BE Mk_o^2.

ON THE FOLLOWING PAGE, ELEMENTS I AND II GENERATE SIMPLE CYLINDERS (SOLID AND HOLLOW). FOR EACH, THE MOMENT OF INERTIA IS CALCULATED DIRECTLY USING THE REAL RADIAL DIMENSIONS OF THE AREA ELEMENT.

FOR THE MORE COMPLEX AREA ELEMENTS III, IV, AND V (AND OTHERS) k_o MAY BE CLOSELY APPROXIMATED AS r_o, THE RADIAL DISTANCE TO THE AREA ELEMENT CENTER OF GRAVITY. IN THESE CASES, CARE MUST BE TAKEN TO AVOID INTRODUCING SIGNIFICANT ERROR IN THE VALUES DETERMINED FOR MOMENT OF INERTIA. ERRORS MAY BE INSIGNIFICANT FOR SMALL MASSES ANYWHERE, FOR MASSES NEAR THE AXIS OF ROTATION, AND EVEN FOR THIN MASSES NEAR THE RIM.

SEE "MOMENT OF INERTIA ACCURACY: $k_o \simeq r_o$" FOR GUIDANCE IN APPLYING THIS SIMPLIFYING APPROXIMATION.

AREA ELEMENT		SOLID OF REVOLUTION MOMENT OF INERTIA
I.		$I = \frac{1}{2} M r^2$
II.		$I = \frac{1}{2} M (r_1^2 + r_2^2)$
III.		$I = M k_o^2 = M(r + 0.424X)^2$
IV		$I = M k_o^2 = M(r + 0.424X)^2$
V.		$I = M k_o^2 = M(r + \frac{y}{3})^2$

*NOTE: r_o LOCATES THE AREA CENTER OF GRAVITY FOR ELEMENTS III, IV, AND V. K_o FOR THE SOLID OF REVOLUTION IS SLIGHTLY LARGER. FOR r SIGNIFICANTLY GREATER THAN 0.424X OR $y/3$, ERRORS IN K_o AND I ARE NEGLIGIBLE.

MOMENT OF INERTIA ACCURACY:

$$k_o \cong r_o$$

AREA ELEMENT MOMENT OF INERTIA (J) CALCULATIONS ARE CORRECT FOR TYPES I AND II. TYPES III, IV, AND V (AND OTHERS) ARE SUFFICIENTLY ACCURATE ONLY WHERE:

1. MASS (M) FOR THE SOLID OF REVOLUTION IS SMALL; i.e., THE ELEMENT IS THIN RADIALLY OR NARROW AXIALLY, OR BOTH; OR

2. THE RADIUS (r) TO THE BASE OF THE ELEMENT IS SMALL; i.e., THE ELEMENT IS AT OR NEAR THE AXIS OF ROTATION; OR

3. THE RADIUS (r) IS LARGE; i.e. THE ELEMENT IS AT OR NEAR THE RIM.

ANY ONE OR COMBINATION OF THESE GREATLY IMPROVES THE ACCURACY OF THE CALCULATED MOMENT OF INERTIA FOR THE YO-YO.

WHEREVER AN ELEMENT IS SUSPECT, CALCULATE J FOR A COMPARABLE RECTANGULAR ELEMENT IN TWO WAYS:

A. USING $r_1 = r$ AND, WITH y AS THE RADIAL DIMENSION, $r_2 = (r+y)$, CALCULATE $J \propto \frac{1}{2}(r_1^2 + r_2^2)$; J IS CORRECT.

B. USING $r_o = (r + \frac{y}{2})$, CALCULATE $J' \propto r_o^2$; J' WILL BE LESS THAN THE CORRECT VALUE J.

THUS,

$$\%\ \text{ERROR},\ J = \frac{J - J'}{J} \times 100\%$$

THIS ERROR, CALCULATED FOR THE COMPARABLE RECTANGULAR ELEMENT, IS LARGER THAN THE ACTUAL ERROR FOR THE SUSPECT ELEMENT. IN THESE ERROR CALCULATIONS, MASS (M) IS NOT A FACTOR.

FOR REFERENCE

IN A TYPICAL YO-YO WHERE $\mathcal{I}$ AND M ARE 20000 kg-m² AND 50×10^{-3} kg RESPECTIVELY:

$$k_0 = \left(\frac{\mathcal{I}}{M}\right)^{\frac{1}{2}} = \left(\frac{20000\times10^{-9}}{50\times10^{-3}}\right)^{\frac{1}{2}} = (400\times10^{-6})^{\frac{1}{2}}$$
$$= 20.00\times10^{-3}\,m/25.4\times10^{-3}\,m/in$$
$$\underline{\underline{k_0 = 0.787\ in}}$$

ASSUMING A 2% ERROR IN THE COMPOSITE CALCULATED $\mathcal{I}'$ FOR THIS YO-YO ($\mathcal{I}$ = 19600)

$$k_0' = \left(\frac{\mathcal{I}'}{M}\right)^{\frac{1}{2}} = \left(\frac{19600\times10^{-9}}{50\times10^{-3}}\right)^{\frac{1}{2}} = (392\times10^{-6})^{\frac{1}{2}}$$
$$= 19.80\times10^{-3}/25.4\times10^{-3}$$
$$\underline{\underline{k_0' = 0.780\ in.}}$$

FOR THE 2% ERROR IN $\mathcal{I}$ THE ERROR IN THE TARGETED RADIUS OF GYRATION IS

$$\%\text{ERROR}, k_0 = \frac{k_0 - k_0'}{k_0}\times100\%$$
$$= \frac{0.007}{0.787}\times100\%$$
$$\underline{\underline{\%\text{ERROR}, k_0 = 0.89\%}} < 1\%$$

UNITS OF MEASURE

IN THE THREE YO-YO STUDIES ON THE FOLLOWING PAGES:

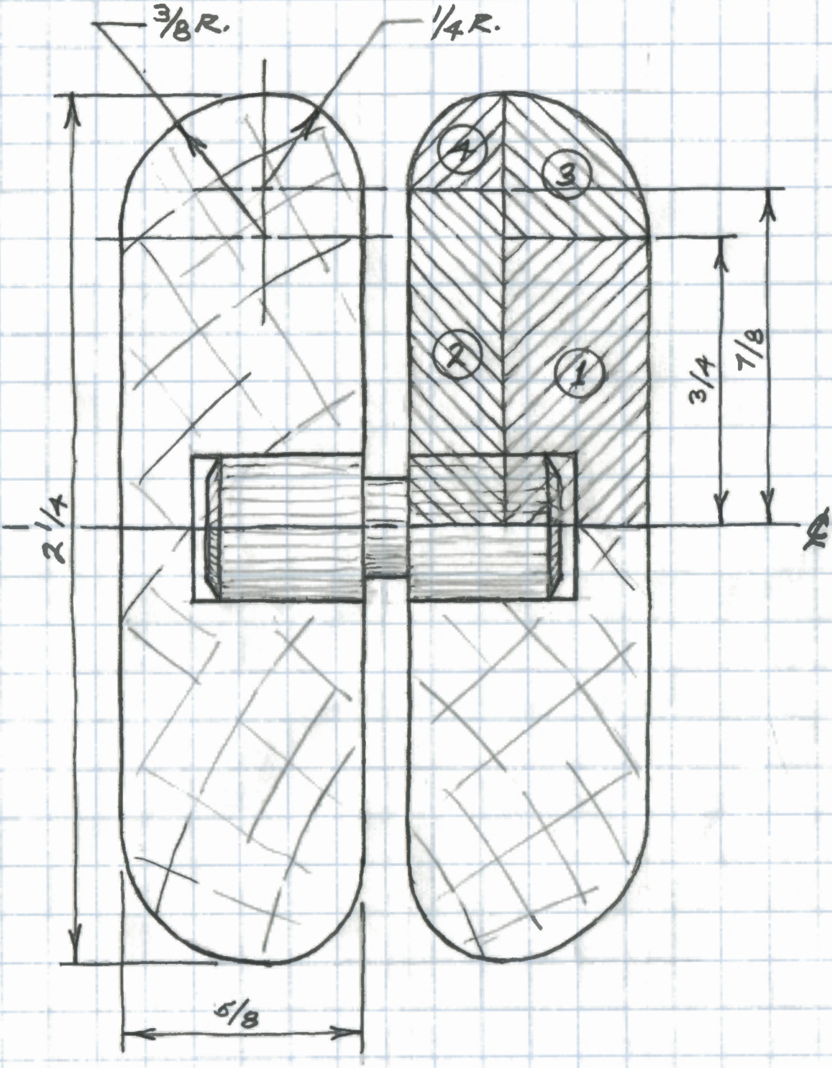
- VOLUME (V) IS FIRST FOUND IN in^3, THEN WEIGHT OR MASS (M) IS DETERMINED FROM MATERIAL DENSITY* IN gm/in^3.
- MOMENT OF INERTIA (I) IS DETERMINED IN $kg\text{-}m^2$, WITH $1\,gm = 1\times10^{-3}\,kg$ AND $1\,in = 25.4\times10^{-3}\,m$.
- RADIUS OF GYRATION (k_o) FOR EACH YO-YO IS FIRST FOUND IN m, THEN GIVEN IN in WITH $1\,m = 1/(25.4\times10^{-3}\,m/in)$.

DENSITIES

MATERIAL	REFERENCE VALUE	CONVERSION	DENSITY (gm/in^3)
MAPLE/BIRCH	44	$\frac{lb}{ft^3}\times\frac{1\,ft^3}{1728\,in^3}\times\frac{453.6\,gm}{lb} =$	12
PLASTIC	75		20
ALUMINUM	0.098	$\frac{lb.}{in^3}\times\frac{453.6\,gm}{lb} =$	44
STEEL	0.283		128
BRASS	0.300		136

* NOTE: ALL NON-METALLIC DENSITIES ARE APPROXIMATE. WHERE POSSIBLE, CHECK BY WEIGHING THE PART.

TRADITIONAL YO-YO



TRADITIONAL
YO-YO PROFILE

TRADITIONAL YO-YO PROFILE

YO-YO ELEMENT	VOLUME	WEIGHT (@ $12 gm/in^3$)
1	$V = (x \cdot y) \cdot 2\pi \cdot \frac{y}{2}$ $= (0.375 \cdot 0.750) \cdot 2\pi \cdot \frac{0.750}{2}$ $V = 0.663\ in^3$	$\times 12 = 7.96$
2	$V = (x \cdot y) \cdot 2\pi \cdot \frac{y}{2}$ $= (0.250 \cdot 0.875) \cdot 2\pi \cdot \frac{0.875}{2}$ $V = 0.601\ in^3$	$\times 12 = 7.21$
3	$V = \left(\frac{\pi x^2}{4}\right) \cdot 2\pi \cdot (r + 0.424x)$ $= \frac{\pi \cdot 0.375^2}{4} \cdot 2\pi \cdot (0.750 + (0.424 \cdot 0.375))$ $= 0.631\ in^3$	$\times 12 = 7.57$
4	$V = \left(\frac{\pi x^2}{4}\right) \cdot 2\pi \cdot (r + 0.424x)$ $= \frac{\pi \cdot 0.250^2}{4} \cdot 2\pi \cdot (0.875 + (0.424 \cdot 0.250))$ $= 0.303\ in^3$	$\times 12 = 3.64$

YO-YO HALF, M = 26.38 gm

YO-YO, M = 52.8 gm

NOTE: THE YO-YO SIDES ARE OF MAPLE AND THE AXLE IS OF BIRCH; BOTH WOODS HAVE A NOMINAL DENSITY OF $12 gm/in^3$. THE SMALL VOID AT THE AXLE END AND THE REDUCED AXLE DIAMETER IN THE STRING GAP ARE INSIGNIFICANT.

TRADITIONAL YO-YO PROFILE
I AND k_o CALCULATION

YO-YO ELEMENT	MOMENT OF INERTIA	I ($kg\text{-}m^2 \times 10^{-9}$)
1	$I = \frac{1}{2}Mr^2,\ M = 7.96\times10^{-3}\,kg$ $= \frac{7.96\times10^{-3}}{2}\cdot(0.750\cdot25.4\times10^{-3})^2$ $= 1444\times10^{-9}\ kg\text{-}m^2$	1444
2	$I = \frac{1}{2}Mr^2,\ M = 7.21\times10^{-3}\,kg$ $= \frac{7.21\times10^{-3}}{2}\cdot(0.875\cdot25.4\times10^{-3})^2$ $= 1781\times10^{-9}\ kg\text{-}m^2$	1781
3	$I = Mk_o^2 = M(r+0.424x)^2$ $= (7.57\times10^{-3})\cdot((0.750+(0.424\cdot0.375))\cdot25.4\times10^{-3})^2$ $= 4035\times10^{-9}\ kg\text{-}m^2$	4035
4	$I = Mk_o^2 = M(r+0.424x)^2$ $= (3.64\times10^{-3})\cdot((0.875+(0.424\cdot0.250))\cdot25.4\times10^{-3})^2$ $= 2260\times10^{-9}\ kg\text{-}m^2$	2260

YO-YO HALF, I = 9520

YO-YO, I = 19040 $kg\text{-}m^2\times10^{-9}$

RADIUS OF GYRATION, $k_o = \left(\frac{I}{M}\right)^{\frac{1}{2}}$

$$= (19040\times10^{-9}/52.8\times10^{-3})^{\frac{1}{2}}$$

$$= 19.0\times10^{-3}\,m \times \frac{1\,in}{25.4\times10^{-3}\,m}$$

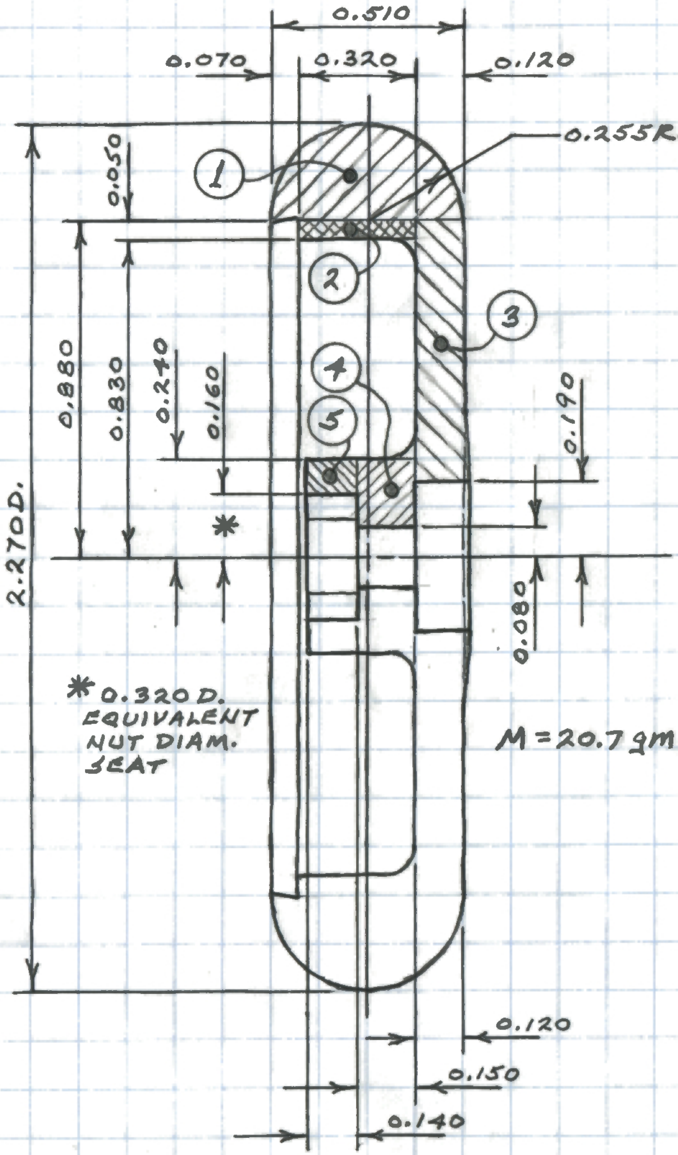
$$k_o = 0.75\ in$$

WHEN YOU CAN MEASURE WHAT YOU ARE SPEAKING ABOUT AND EXPRESS IT IN NUMBERS YOU KNOW SOMETHING ABOUT IT, BUT WHEN YOU CANNOT EXPRESS IT IN NUMBERS YOUR KNOWLEDGE OF IT IS OF A MEAGRE AND UNSATISFACTORY KIND.

WILLIAM THOMSON, LORD KELVIN
1824 – 1907

* * *

CONTEMPORARY YO-YO



RIM-WEIGHTED (PLASTIC)
YO-YO PROFILE

CONTEMPORARY YO-YO PROFILE

YO-YO ELEMENT	VOLUME	WEIGHT (@ 20 gm/in³)
1	$V = \left(\frac{\pi x^2}{2}\right) \cdot 2\pi \cdot (r + 0.424x)$ $= \frac{\pi \cdot 0.255^2}{2} \cdot 2\pi \cdot (0.880 + (0.424 \cdot 0.255))$ $V = 0.634\ in^3$	$\times 20 = 12.68$
2	$V = (x \cdot y) \cdot 2\pi \cdot (r + \frac{y}{2})$ $= (0.320 \cdot 0.050) \cdot 2\pi \cdot (0.830 + \frac{0.050}{2})$ $V = 0.086\ in^3$	$\times 20 = 1.72$
3	$V = (x \cdot y) \cdot 2\pi \cdot (r + \frac{y}{2})$ $= (0.120 \cdot 0.690) \cdot 2\pi \cdot (0.190 + \frac{0.690}{2})$ $V = 0.278\ in^3$	$\times 20 = 5.56$
4	$V = (x \cdot y) \cdot 2\pi \cdot (r + \frac{y}{2})$ $= (0.150 \cdot 0.160) \cdot 2\pi \cdot (0.080 + \frac{0.160}{2})$ $V = 0.024\ in^3$	$\times 20 = 0.48$
5	$V = (x \cdot y) \cdot 2\pi \cdot (r + \frac{y}{2})$ $= (0.140 \cdot 0.080) \cdot 2\pi \cdot (0.160 + \frac{0.080}{2})$ $V = 0.014\ in^3$	$\times 20 = 0.28$

YO-YO HALF, M = 20.72 gm

YO-YO HALVES, M = 41.4 gm

NOTE: THE YO-YO HALVES PLASTIC MATERIAL HAS A KNOWN NOMINAL DENSITY OF 20 gm/in³.

CONTEMPORARY YO-YO PROFILE
I AND K_o CALCULATION

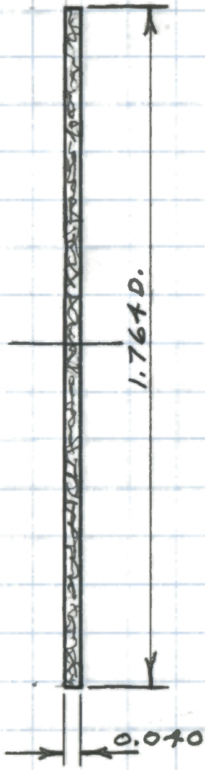
YO-YO ELEMENT	MOMENT OF INERTIA	I ($kg\text{-}m^2 \times 10^{-9}$)
1	$I = Mk_o^2 = M(r + 0.424x)^2, M = 12.68 \times 10^{-3} kg$ $= 12.68 \times 10^{-3} \cdot ((0.880 + (0.424 \cdot 0.255)) \cdot 25.4 \times 10^{-3})^2$ $I = 7987 \times 10^{-9} kg\text{-}m^2$	7987
2	$I = \frac{1}{2} M(r_1^2 + r_2^2), M = 1.72 \times 10^{-3} kg$ $= \frac{1.72 \times 10^{-3}}{2} \cdot (0.830^2 + 0.880^2) \cdot (25.4 \times 10^{-3})^2$ $= 812 \times 10^{-9} kg\text{-}m^2$	812
3	$I = \frac{1}{2} M(r_1^2 + r_2^2), M = 5.56 \times 10^{-3} kg$ $= \frac{5.56 \times 10^{-3}}{2} \cdot (0.190^2 + 0.880^2) \cdot (25.4 \times 10^{-3})^2$ $= 1454 \times 10^{-9} kg\text{-}m^2$	1454
4	$I = \frac{1}{2} M(r_1^2 + r_2^2), M = 0.48 \times 10^{-3} kg$ $= \frac{0.48 \times 10^{-3}}{2} \cdot (0.080^2 + 0.240^2) \cdot (25.4 \times 10^{-3})^2$ $= 10 \times 10^{-9} kg\text{-}m^2$	10
5	$I = \frac{1}{2} M(r_1^2 + r_2^2), M = 0.28 \times 10^{-3} kg$ $= \frac{0.28 \times 10^{-3}}{2} \cdot (0.160^2 + 0.240^2) \cdot (25.4 \times 10^{-3})^2$ $= 8 \times 10^{-9} kg\text{-}m^2$	8

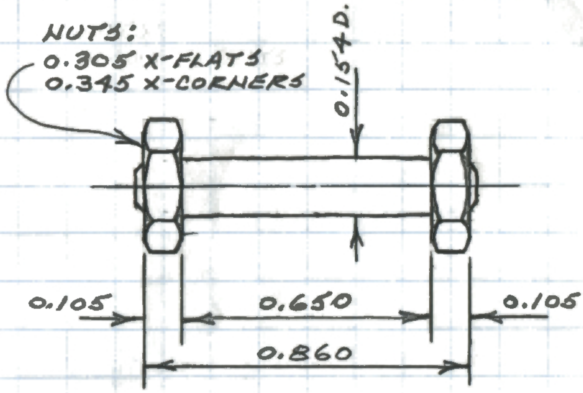
YO-YO HALF, I = 10271

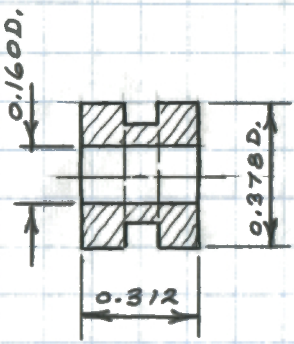
YO-YO HALVES, I = 20542*

$kg\text{-}m^2 \times 10^{-9}$

* SEE PAGE 24.







YO-YO INSERT
(CARD, M = 2.0 gm)

AXLE SCREW & NUTS
(STEEL, M = 3.7 gm)
AND
AXLE SLEEVE
(BIRCH, M = 0.3 gm)

COMPONENT	WEIGHT
YO-YO HALVES	41.4
INSERTS	4.0
AXLE SCREW & NUTS	3.7
AXLE SLEEVE	0.3
YO-YO TOTAL, M =	49.4 gm

YO-YO HALVES 20542

INSERTS (2 CARDS, $M = 4.0\,gm$)

$$I = \frac{1}{2} M r^2$$
$$= \frac{4.0 \times 10^{-3}}{2} \cdot (0.882 \cdot 25.4 \times 10^{-3})^2$$
$$I = 1004 \times 10^{-9}\ kg\text{-}m^2 \qquad 1004$$

AXLE SCREW (STEEL, $M = 2.0\,gm$)

$$I = \frac{1}{2} M r^2$$
$$= \frac{2.0 \times 10^{-3}}{2} \cdot (0.770 \cdot 25.4 \times 10^{-3})^2$$
$$I = 4 \times 10^{-9}\ kg\text{-}m^2 \qquad 4$$

NUTS (STEEL, $M = 1.7\,gm$)

$$I = \frac{1}{2} M (r_1^2 + r_2^2)$$
$$= \frac{1.7 \times 10^{-3}}{2} \cdot (0.077^2 + 0.160^2) \cdot (25.4 \times 10^{-3})^2$$
$$I = 17 \times 10^{-9}\ kg\text{-}m^2 \qquad 17$$

AXLE SLEEVE (BIRCH, $M = 0.3\,gm$)

$$I = \frac{1}{2} M (r_1^2 + r_2^2)$$
$$= \frac{0.3 \times 10^{-3}}{2} \cdot (0.080^2 + 0.189^2) \cdot (25.4 \times 10^{-3})^2$$
$$I = 4 \times 10^{-9}\ kg\text{-}m^2 \qquad 4$$

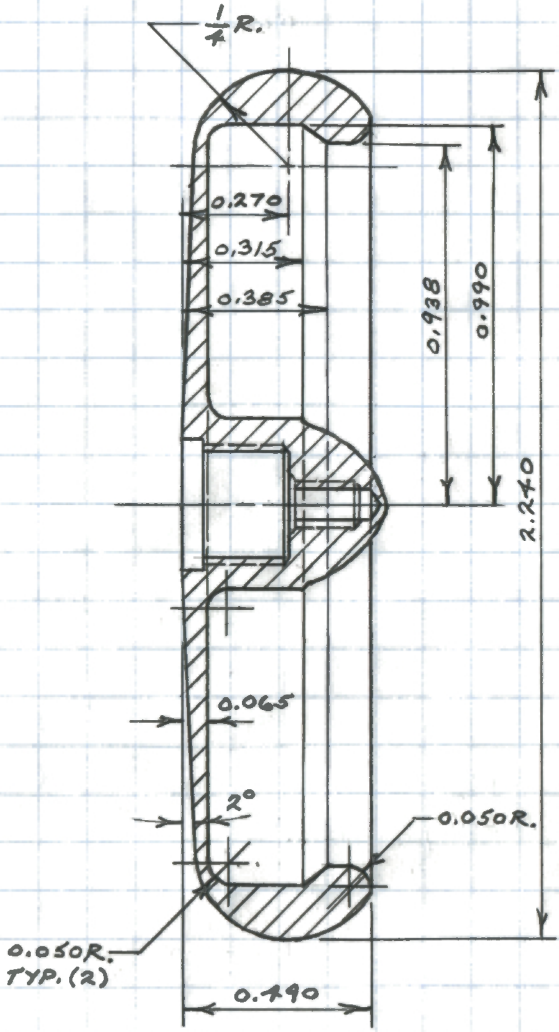
YO-YO TOTAL, $I = 21571$ $kg\text{-}m^2 \times 10^{-9}$

RADIUS OF GYRATION, $K_o = \left(\frac{I}{M}\right)^{\frac{1}{2}}$

$$= (21571 \times 10^{-9} / 49.4 \times 10^{-3})^{\frac{1}{2}}$$
$$= 20.9 \times 10^{-3} m \times \frac{1\,in}{25.4 \times 10^{-3} m}$$
$$K_o = 0.82\,in$$

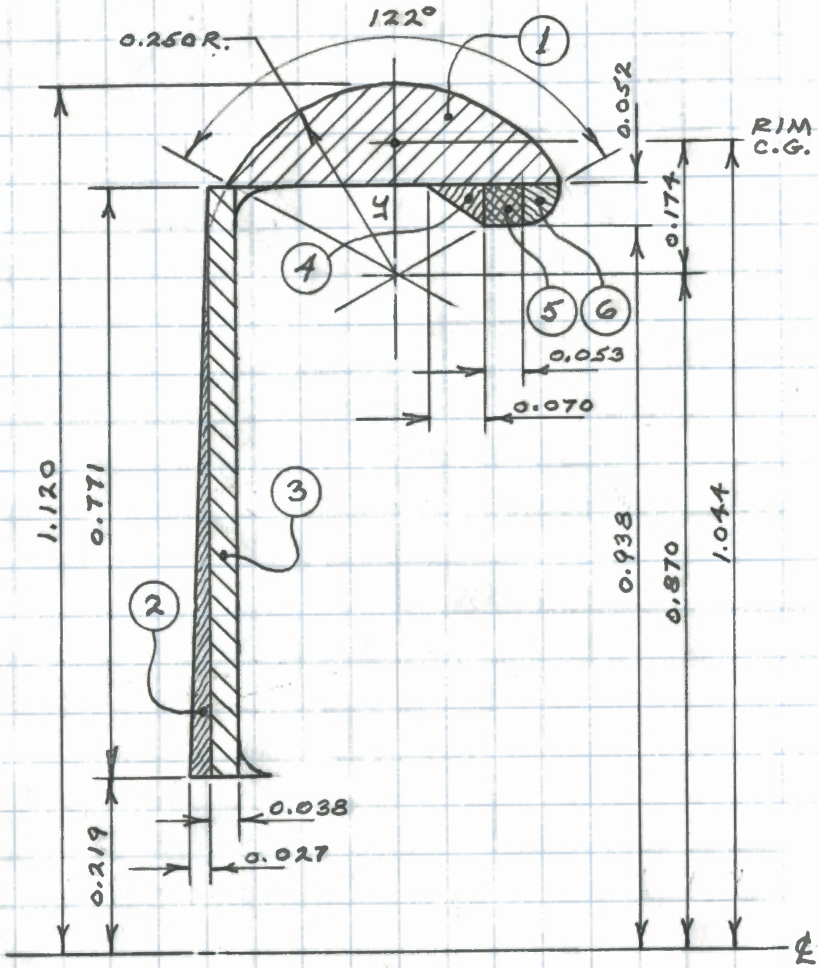
MODERN TRANSAXLE YO-YO

NOTE: SEE NEXT PAGES FOR FURTHER HUB AND RIM DETAIL.



MODERN TRANSAXLE
Yo-Yo PROFILE*

*ALUMINUM



1 RIM ELEMENT; SEE PAGE 35, REFERENCE NO. 5 AND PAGE 58 THEREIN FOR THESE DERIVATIONS:

$$a = \frac{1}{2} R^2 (0.01745\,a - \sin a)$$
$$= \frac{1}{2} \cdot 0.250^2 \cdot ((0.01745 \cdot 122) - \sin 122)$$
$$a = 0.040 \text{ in}^2$$

$$y = \frac{2}{3} \cdot \frac{R^3 \sin^3 0.5a}{A}$$
$$= \frac{2}{3} \cdot \frac{0.250^3 \cdot \sin^3 61}{0.040}$$
$$y = 0.174 \text{ in}$$

RIM, WEB, AND LIP ELEMENTS

RIM, WEB, AND LIP ELEMENTS

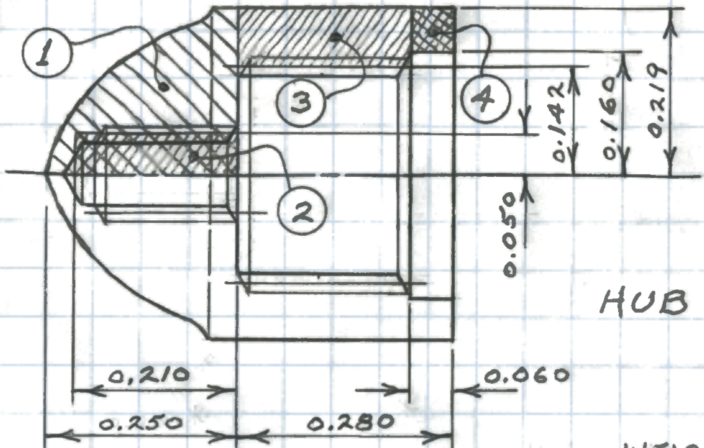
YO-YO ELEMENT	VOLUME	WEIGHT (@ 4.4 gm/in³)
1	$V = a \cdot 2\pi \cdot R_0$ $= 0.040 \cdot 2\pi \cdot (0.870 + 0.174)$ $= 0.262 \text{ in}^3$	$\times 44 = 11.53$
2	$V = \left(\frac{x \cdot y}{2}\right) \cdot 2\pi \cdot \left(r + \frac{y}{3}\right)$ $= \frac{0.027 \cdot 0.771}{2} \cdot 2\pi \cdot \left(0.219 + \frac{0.771}{3}\right)$ $= 0.031 \text{ in}^3$	$\times 44 = 1.36$
3	$V = (x \cdot y) \cdot 2\pi \cdot \left(r + \frac{y}{2}\right)$ $= 0.038 \cdot 0.771 \cdot 2\pi \cdot \left(0.219 + \frac{0.771}{2}\right)$ $= 0.111 \text{ in}^3$	$\times 44 = 4.88$
4	$V = \left(\frac{x \cdot y}{2}\right) \cdot 2\pi \cdot \left(r + \frac{2y}{3}\right)$ $= \frac{0.070 \cdot 0.052}{2} \cdot 2\pi \cdot \left(0.938 + \frac{2 \cdot 0.052}{3}\right)$ $= 0.011 \text{ in}^3$	$\times 44 = 0.48$
5	$V = (x \cdot y) \cdot 2\pi \cdot \left(r + \frac{y}{2}\right)$ $= 0.053 \cdot 0.052 \cdot 2\pi \cdot \left(0.938 + \frac{0.052}{2}\right)$ $= 0.017 \text{ in}^3$	$\times 44 = 0.75$
6	$V = \left(\frac{\pi x^2}{4}\right) \cdot 2\pi \cdot (r + 0.576x)$ $= \frac{\pi \cdot 0.052^2}{4} \cdot 2\pi \cdot (0.938 + (0.576 \cdot 0.052))$ $V = 0.013 \text{ in}^3$	$\times 44 = 0.57$

RIM, WEB, AND LIP ELEMENTS, $\underline{\underline{M = 19.57 \text{ gm}}}$

RIM, WEB, AND LIP ELEMENTS

YO-YO ELEMENT	MOMENT OF INERTIA	I ($kg\text{-}m^2 \times 10^{-9}$)
1	$I = M k_o^2,\ k_o = 1.044,\ M = 11.53\ gm$ $= 11.53\times10^{-3}\cdot(1.044\cdot 25.4\times10^{-3})^2$ $I = 8108\times10^{-9}\ kg\text{-}m^2$	8108
2	$I = M k_o^2 \simeq M(r+\frac{y}{3})^2,\ M = 1.36\ gm$ $= 1.36\times10^{-3}\cdot((0.219+\frac{0.771}{3})\cdot(25.4\times10^{-3}))^2$ $I = 199\times10^{-9}\ kg\text{-}m^2$	199
3	$I = \frac{1}{2}M(r_1^2+r_2^2),\ M = 4.88\ gm$ $= \frac{4.88\times10^{-3}}{2}\cdot(0.219^2+0.490^2)\cdot(25.4\times10^{-3})^2$ $= 1618\times10^{-9}\ kg\text{-}m^2$	1618
4	$I = M k_o^2 \simeq M(r+\frac{2y}{3})^2,\ M = 0.48\ gm$ $= 0.48\times10^{-3}\cdot((0.938+\frac{2\cdot0.052}{3})\cdot(25.4\times10^{-3}))^2$ $I = 293\times10^{-9}\ kg\text{-}m^2$	293
5	$I = \frac{1}{2}M(r_1^2+r_2^2),\ M = 0.75\ gm$ $= \frac{0.75\times10^{-3}}{2}\cdot(0.938^2+0.990^2)\cdot(25.4\times10^{-3})^2$ $I = 450\times10^{-9}\ kg\text{-}m^2$	450
6	$I = M k_o^2 \simeq M(r+0.576\,y)^2,\ M = 0.57\ gm$ $= 0.57\times10^{-3}\cdot((0.938+(0.576\cdot0.052))\cdot 25.4\times10^{-3})^2$ $I = 345$	345

RIM, WEB, AND LIP ELEMENTS, $I = 11013$ $kg\text{-}m^2\times10^{-9}$



HUB

ELEMENT	VOLUME	WEIGHT (@44gm/in^3)
1	$a = \frac{2}{3} x y$, $k_0 = \frac{3}{8} y$ $V = \frac{2}{3} \cdot (x \cdot y) \cdot 2\pi \cdot \frac{3}{8} \cdot y$ $= \frac{2}{3} \cdot (0.250 \cdot 0.219) \cdot 2\pi \cdot \frac{3}{8} \cdot 0.219$ $V = 0.019\ in^3$	$\times 44 =$ 0.84
2	$V = (x \cdot y) \cdot 2\pi \cdot \frac{y}{2}$ $= 0.210 \cdot 0.050 \cdot 2\pi \cdot \frac{0.050}{2}$ $V = 0.002\ in^3$	$\times 44 =$ (0.09)
3	$V = (x \cdot y) \cdot 2\pi \cdot (r + \frac{y}{2})$ $= 0.220 \cdot 0.077 \cdot 2\pi \cdot (0.142 + \frac{0.077}{2})$ $V = 0.019\ in^3$	$\times 44 =$ 0.85
4	$V = (x \cdot y) \cdot 2\pi \cdot (r + \frac{y}{2})$ $= 0.060 \cdot 0.059 \cdot 2\pi \cdot (0.160 + \frac{0.059}{2})$ $V = 0.004\ in^3$	$\times 44 =$ 0.18

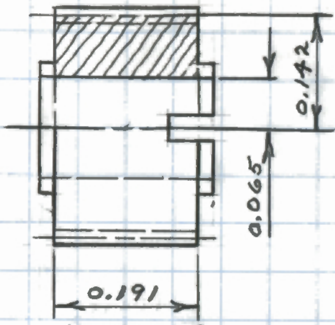
HUB, M = 1.78 gm

HUB

YO-YO ELEMENT	MOMENT OF INERTIA	I $(kg\text{-}m^2 \times 10^{-9})$
1	$I = Mk_o^2 \simeq M\left(\frac{3y}{8}\right)^2,\ M = 0.84\,gm$	
	$= 0.84\times10^{-3}\cdot\left(\frac{3\cdot0.219}{8}\cdot25.4\times10^{-3}\right)^2$	
	$I = 3.66\times10^{-9}\ kg\text{-}m^2$	3.66
2	$I = \frac{1}{2}Mr^2,\ M = 0.09\,gm$	
	$= \frac{0.09\times10^{-3}}{2}\cdot(0.050\cdot25.4\times10^{-3})^2$	
	$I = 0.07\times10^{-9}\ kg\text{-}m^2$	(0.07)
3	$I = \frac{1}{2}M(r_1^2+r_2^2),\ M = 0.85\,gm$	
	$= \frac{0.85\times10^{-3}}{2}\cdot(0.142^2+0.219^2)\cdot(25.4\times10^{-3})^2$	
	$I = 18.68\times10^{-9}\ kg\text{-}m^2$	18.68
4	$I = \frac{1}{2}M(r_1^2+r_2^2),\ M = 0.18\,gm$	
	$= \frac{0.18\times10^{-3}}{2}\cdot(0.160^2+0.219^2)\cdot(25.4\times10^{-3})^2$	
	$I = 4.27\times10^{-9}\ kg\text{-}m^2$	4.27
		26.54

HUB, I = 27.0 $(kg\text{-}m^2\times10^{-9})$

TRANSAXLE PARTS



4X SCALE

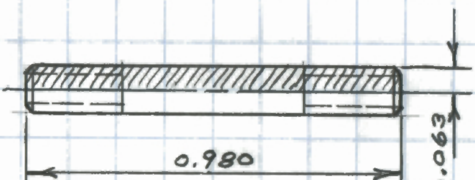
GAP SCREW (2)

$M = 1.30\,gm$ (EACH)

$$J = \frac{1}{2} M (r_1^2 + r_2^2)$$

$$= \frac{1.30 \times 10^{-3}}{2} \cdot (0.065^2 + 0.142^2) \cdot (25.4 \times 10^{-3})^2$$

$$J = 10 \times 10^{-9}\,kg\text{-}m^2$$



2X SCALE

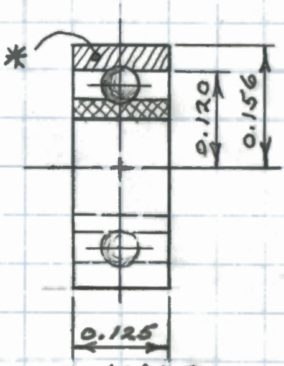
AXLE SCREW

$M = 1.30\,gm$

$$J = \frac{1}{2} M r^2$$

$$= \frac{1.30 \times 10^{-3}}{2} \cdot (0.063^2) \cdot (25.4 \times 10^{-3})^2$$

$$J = 2 \times 10^{-9}\,kg\text{-}m^2$$



4X SCALE

BALL-BEARING

*$M = 0.40\,gm$

$$J = \frac{1}{2} M (r_1^2 + r_2^2)$$

$$= \frac{0.40 \times 10^{-3}}{2} \cdot (0.120^2 + 0.156^2) \cdot (25.4 \times 10^{-3})^2$$

$$J = 5 \times 10^{-9}\,kg\text{-}m^2$$

*NOTE: OUTER RACE ROTATES ONLY IN STRING UNWIND AND REWIND.

MODERN TRANSAXLE YO-YO
M, I, AND k_o SUMMARY

	M ($kg \times 10^{-3}$)	I ($kg\text{-}m^2 \times 10^{-9}$)
YO-YO HALVES		
RIM (2)	23.06	16216
WEB (2)	12.48	3634
LIP (2)	3.60	2176
HUB (2)	3.56	54*
TRANSAXLE PARTS		
GAP SCREW (2)	2.60	20*
AXLE SCREW	1.30	2*
BALL-BEARING	0.70	5*
TOTAL:	47.30	22107

$$\underline{\underline{I = 22107 \times 10^{-9}\, kg\text{-}m^2}}$$

$$k_o = \left(\frac{I}{M}\right)^{\frac{1}{2}} = \left(\frac{22107 \times 10^{-9}}{47.3 \times 10^{-3}}\right)^{\frac{1}{2}} = (467 \times 10^{-6})^{\frac{1}{2}}$$
$$= 21.6 \times 10^{-3}\, m \times 1\, in / 25.4 \times 10^{-3}\, m$$

$$\underline{\underline{k_o = 0.85\, in}}$$

*k_o ERROR FOR IGNORING THESE I VALUES IS NEGLIGIBLE; FOR IGNORING THE COMPANION M VALUES IS OVER 10%!

AUTHOR'S NOTE

AFICIONADOS MAY RECOGNIZE THE YO-YOS ANALYZED HERE.

THE TRADITIONAL YO-YO HAS ITS EARLY ROOTS IN ANTIQUITY. THE EMBODIMENT HERE MOST CLOSELY APPROXIMATES THE WHAT'S NEXT INC. SOLID AND LAMINATED MAPLE YO-YOS IN CURRENT PRODUCTION.

THE CONTEMPORARY YO-YO ANALYSIS USES THE SPINTASTICS "TECHNIC" MODEL; A POPULAR STYLE IN TODAY'S MARKET.

THE MODERN TRANSAXLE YO-YO HERE IS THE TOM KUHN SILVER BULLET 2.

THE AUTHOR HAS HAD NO PART IN THE DESIGN OR PRODUCTION OF THE FIRST TWO ENTRIES. THE SB2 PROFILE IS A TOM KUHN CREATION FIRST PRODUCED IN HIS FAMOUS FIXED AXLE SB1.

THESE THREE STYLES ARE GOOD REPRESENTATIVES OF YO-YOS IN WOOD, PLASTIC, AND ALUMINUM. NO SLIGHT OF THE MANY OTHER FINE PRODUCTS IS INTENDED OR INFERRED.

HAPPY DAYS -

Don Watson

Captain YO

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* * *

CLAIMERS/DISCLAIMERS

1. USING THE APPROXIMATION $k_o \simeq r_o$ (PAGES 10, 11, AND ELSEWHERE) UNDERSTATES THE MOMENT OF INERTIA I ABOUT 2% AND THE RADIUS OF GYRATION k_o LESS THAN 2% FOR EACH OF THE YOYOS ANALYZED. USE OF MORE COMPLEX AND SLIGHTLY MORE ACCURATE INTEGRAL CALCULUS DERIVATIONS IS AVOIDED, AS IS SIGNIFICANTLY GREATER EFFORT.

2. TO MY KNOWLEDGE AND AFTER MUCH CHECKING, CROSS-CHECKING, CORRECTING, AND RECHECKING, THE METHODS AND THE ARITHMETIC PRESENTED ARE CORRECT. MY APOLOGIES TO ANY READER FINDING ANY ERROR(S?), AND MY APPRECIATION IF YOU WILL WRITE TO ME OF ANY FOUND.

3. USERS OF THIS METHOD MUST ASSUME FULL RESPONSIBILITY FOR RESULTS OBTAINED FOR THIS AND OTHER ENGINEERING APPLICATIONS.

4. MEASUREMENT DATA FOR THE YO-YOS CITED HERE MAY DIFFER SIGNIFICANTLY IN THE MARKETED VERSIONS.

— NOTES —